V E R I F I C A T I O N # 5

Angular Velocity

The purpose of this verification is to confirm that the angular velocity algorithm used by the program is working correctly.

The example consists of a slope with two benches and a single rock that begins its travel at the crest of the slope. This example is identical to verification #1 (Projectile) except that angular velocity has been considered in the equations. The rock was given an initial velocity and bounced a number of times before coming to rest at the base of the slope.

The slope was created by making minor modifications to the geometry of an actual slope profile. The geometry was modified so that the impacts would occur on slope segments with a positive slope, a negative slope and a horizontal segment.

The slope geometry and the input parameters were configured so that no sliding would occur. No statistics were incorporated into this verification (i.e. only mean values were used, all standard deviations were set to 0). Although rock trajectories in an actual simulation typically have dozens of steps, only the first four steps are followed here. This was done in the interest of brevity.

The minimum velocity (V_{MIN}) was set to 1 m/s. This minimum velocity was selected so that the simulation did not end before the four steps were complete. Other numbers used in this example (e.g. the mass of the rock) were selected primarily for their ease in manual calculations.

Initial Conditions

The rock starts at location $X_0 = 0$ m, $Y_0 = 60$ m (which coincides with the first slope vertex). The rock was given an initial velocity of $V_{X0} = 7$ m/s, $V_{Y0} = 2$ m/s and a mass of 10 Kg. The rock was given an angular velocity $\omega_0 = 0$ m/s. The rock is assumed to be a sphere with a density of 2100 kg/m^3 for purposes of calculating a radius and moment of inertia from the mass:

$$r = \sqrt[3]{\frac{3m}{4\pi\gamma}} = \sqrt[3]{\frac{3 \cdot 10}{4 \cdot \pi \cdot 2100}} = 0.104\,m$$

$$I = \frac{2mr^2}{5} = \frac{2 \cdot 10 \cdot 0.104^2}{5} = 0.0436\,kgm^2$$

The location of the slope vertices and the coefficients of restitution for each slope segment are presented in the following table (these are the same values as verification #1):

	x co-ordinate	y co-ordinate	R_N	R_T
Vertex 1	0	60		
Segment 1			0.5	0.8
Vertex 2	7	39		
Segment 2			0.5	0.8
Vertex 3	19	40		
Segment 3			0.5	0.8
Vertex 4	26	22		
Segment 4			0.6	0.9
Vertex 5	38	20		
Segment 5			0.6	0.9
Vertex 6	46	0		
Segment 6			0.4	0.6
Vertex 7	89	0		

Table A.5.1 - Slope geometry

A diagram of the rock trajectories with and without considering angular velocity follows:

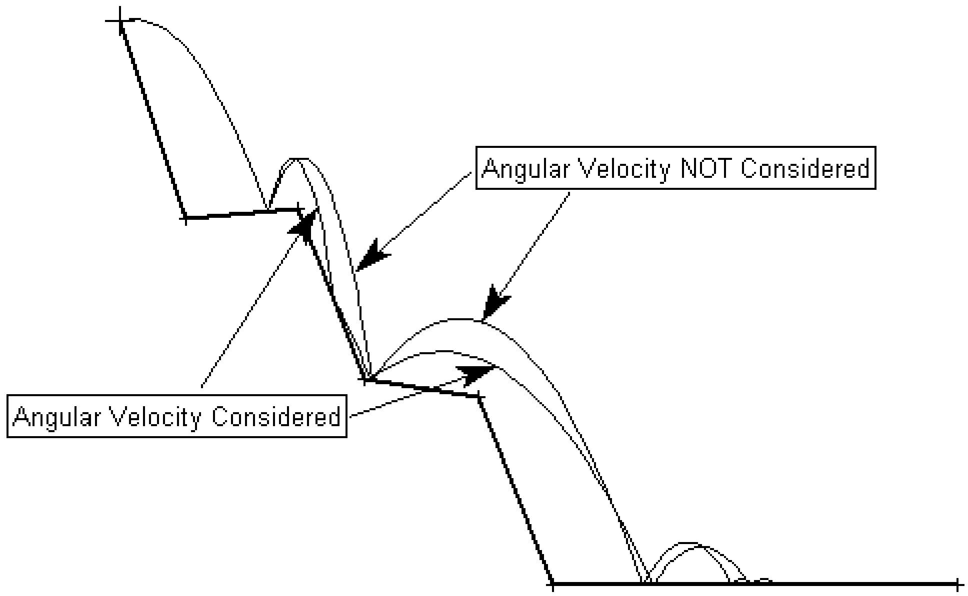

Figure A.5.1 - Rock trajectories in RocFall (with and without angular velocity)

Hand Calculations

The angular velocity calculations are very similar to the projectile calculations. The difference between the two is between the steps where the velocity is transformed from horizontal and vertical components into normal and tangential components; and when it is transformed back into horizontal and vertical components. For a more detailed explanation of the steps involved in the projectile algorithm, please consult verification #1.

Step 1

The rock starts at location $X_0 = 0$ m, $Y_0 = 60$ m (which coincides with the first slope vertex). The rock was given an initial velocity of $V_{X0} = 7$ m/s, $V_{Y0} = 2$ m/s. The necessary parameters are determined and the quadratic equation is solved to find the time of intersection with the second slope segment (using equations 4.8, 4.9, 4.10, 4.11 and 4.12):

$$q = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(40-39)}{(19-7)} \cong 0.0833$$

$$a = \frac{1}{2} g \cong -4.90$$

$$b = V_{Y0} - qV_{X0} = 2 - (0.0833)7 \cong 1.417$$

$$c = Y_0 - Y_1 + q(X_1 - X_0) = 60 - 39 + 0.0833(7 - 0) \cong 21.58$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(1.417) \pm \sqrt{(1.417)^2 - 4(-4.90)(21.58)}}{2(-4.90)} \cong -1.958 \; or \; 2.25 \; s$$

t = -1.958 s is rejected because t must lie in the range [0,∞]. The intersection point and pre-intersection velocity are found by substituting t back into equations 4.3, 4.4, 4.5 and 4.6:

$$X_I = V_{X0} t + X_0 = 7(2.25) + 0 = 15.729 \; m$$

$$Y_I = \frac{1}{2} g t^2 + V_{Y0} t + Y_0 = \frac{1}{2}(-9.81)(2.25)^2 + 2(2.25) + 60 = 39.727 \; m$$

$$V_{XB} = V_{X0} = 7 \; m/s$$

$$V_{YB} = V_{Y0} + gt = 2 + (-9.81)2.25 = -20.04 \; m/s$$

The velocities are transformed into components normal and tangential to the slope segment (using equations 4.13 and 4.14):

$$\theta = \tan^{-1}(q) = 4.77^0$$

$$V_{NB} = (V_{YB})\cos(\theta) - (V_{XB})\sin(\theta) = (-20.04)\cos(4.77) - (7)\sin(4.77) = -20.6 \; m/s$$

$$V_{TB} = (V_{YB})\sin(\theta) + (V_{XB})\cos(\theta) = (-20.04)\sin(4.77) + (7)\cos(4.77) = 5.31 \; m/s$$

The friction function (F_1) and scaling function (F_2) are calculated (the empirical constants C_{F1} = 6.096 m/s (20 ft/s) and C_{F2} = 76.2 m/s (250 ft/s) are used in these equations):

$$F_1 = R_T + \frac{(1 - R_T)}{\left(\frac{(V_{TB} - \omega_B \cdot r)}{C_{F1}}\right)^2 + 1.2} = 0.8 + \frac{(1 - 0.8)}{\left(\frac{(5.31 - 0 \cdot (0.104))}{6.096}\right)^2 + 1.2} = 0.9021$$

$$F_2 = \frac{R_T}{[\frac{V_{NB}}{C_{F2} \cdot R_N}]^2 + 1} = \frac{0.8}{[\frac{-20.6}{76.2 \cdot 0.5}]^2 + 1} = 0.620$$

Calculate the outgoing velocities:

$$V_{NA} = R_N V_{NB} = 0.5(-20.6) = -10.28\ m/s$$

$$V_{TA} = \sqrt{\frac{r^2[I(\omega_B)^2 + m(V_{TB})^2]\cdot F_1 \cdot F_2}{I + m\cdot r^2}}$$

$$= \sqrt{\frac{(0.104)^2[0.0436(0)^2 + 10(5.31)^2]\cdot 0.9021\cdot 0.620}{0.0436 + 10\cdot(0.104)^2}} = 3.36\, m/s$$

$$\omega_A = \frac{V_{TA}}{r} = \frac{3.36}{0.104} = 32.16\, rad/s$$

The velocities are transformed back into vertical and horizontal components (using equations 4.17 and 4.18):

$$V_{XA} = (V_{NA})\sin(\theta) + (V_{TA})\cos(\theta) = (-10.28)\sin(4.77) + (3.36)\cos(4.77) = 2.49\ m/s$$

$$V_{YA} = (V_{TA})\sin(\theta) - (V_{NA})\cos(\theta) = (3.36)\sin(4.77) - (-10.28)\cos(4.77) = 10.52\ m/s$$

Step 1 is complete. The velocity of the rock, after impact, is calculated:

$$V_{CHECK} = \sqrt{V_{XA}^2 + V_{YA}^2} = \sqrt{(2.49)^2 + (10.52)^2} = 10.81\ m/s$$

Since the velocity of the rock, V_{CHECK} (= 10.81 m/s) is greater than the minimum velocity, V_{MIN} (= 1.0 m/s), the rock is still considered to be moving. Since the rock is still moving, the simulation must continue for at least one more step.

Step 2

The final rock conditions for step 1 are used as the initial conditions for step 2. That is:

$$X_{0(step2)} = X_{I(step1)}$$

$$Y_{0(step2)} = Y_{I(step1)}$$

$$V_{X0(step2)} = V_{XA(step1)}$$

$$V_{Y0(step2)} = V_{YA(step1)}$$

$$\omega_{0(step2)} = \omega_{A(step1)}$$

The necessary parameters are determined and the quadratic equation is solved to find the time to intersection with the third slope segment:

$$q = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(22-40)}{(26-19)} \cong -2.5714$$

$$a = \frac{1}{2}g \cong -4.90$$

$$b = V_{Y0} - qV_{X0} = 10.52 - (-2.571)2.49 \cong 16.93$$

$$c = Y_0 - Y_1 + q(X_1 - X_0) = 39.727 - 40 + (-2.5714)(19 - 15.729) \cong -8.683$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(16.90) \pm \sqrt{(16.90)^2 - 4(-4.90)(-8.683)}}{2(-4.90)} \cong 0.627 \; or \; 2.824 \; s$$

$t =$ 0.627 s is rejected because the slope is not defined at this time. The intersection point and pre-impact velocity are determined:

$$X_I = V_{X0}t + X_0 = 2.49(2.824) + 15.729 = 22.764 \; m$$

$$Y_I = \frac{1}{2}gt^2 + V_{Y0}t + Y_0 = \frac{1}{2}(-9.81)(2.824)^2 + 10.52(2.824) + 39.727 = 30.322 \; m$$

$$V_{XB} = V_{X0} = 2.49 \; m/s$$

$$V_{YB} = V_{Y0} + gt = 10.52 + (-9.81)2.824 = -17.18 \; m/s$$

The velocities are transformed into components normal and tangential to the slope segment:

$$\theta = \tan^{-1}(q) = -68.75^0$$

$$V_{NB} = (V_{YB})\cos(\theta) - (V_{XB})\sin(\theta) = (-17.18)\cos(-68.75) - (2.49)\sin(-68.75)$$
$$= -3.90\ m/s$$
$$V_{TB} = (V_{YB})\sin(\theta) + (V_{XB})\cos(\theta) = (-17.18)\sin(-68.75) + (2.49)\cos(-68.75)$$
$$= 16.92\ m/s$$

The friction function (F_1) and scaling function (F_2) are calculated:

$$F_1 = R_T + \frac{(1-R_T)}{\left(\frac{(V_{TB} - \omega_B \cdot r)}{C_{F1}}\right)^2 + 1.2} = 0.8 + \frac{(1-0.8)}{\left(\frac{(16.92 - 32.16\cdot(0.104))}{6.096}\right)^2 + 1.2} = 0.8325$$

$$F_2 = \frac{R_T}{[\frac{V_{NB}}{C_{F2}\cdot R_N}]^2 + 1} = \frac{0.8}{[\frac{-3.90}{76.2\cdot 0.5}]^2 + 1} = 0.7917$$

Calculate the outgoing velocities:

$$V_{NA} = R_N V_{NB} = 0.5(-3.90) = -1.953\ m/s$$
$$V_{TA} = \sqrt{\frac{r^2[I(\omega_B)^2 + m(V_{TB})^2]\cdot F_1 \cdot F_2}{I + m\cdot r^2}}$$
$$= \sqrt{\frac{(0.104)^2[0.0436(32.16)^2 + 10(16.92)^2]\cdot 0.8325\cdot 0.7917}{0.0436 + 10\cdot(0.104)^2}} = 11.70\,m/s$$

$$\omega_A = \frac{V_{TA}}{r} = \frac{11.70}{0.104} = 112.1\,rad/s$$

The velocities are transformed back into vertical and horizontal components:

$$V_{XA} = (V_{NA})\sin(\theta) + (V_{TA})\cos(\theta) = (-1.953)\sin(-68.75) + (11.70)\cos(-68.75)$$
$$= 6.06\ m/s$$
$$V_{YA} = (V_{TA})\sin(\theta) - (V_{NA})\cos(\theta) = (11.70)\sin(-68.75) - (-1.953)\cos(-68.75)$$
$$= -10.19\ m/s$$

Step 2 is complete. The velocity of the rock, after impact, is calculated:

$$V_{CHECK} = \sqrt{V_{XA}^2 + V_{YA}^2} = \sqrt{(6.06)^2 + (-10.19)^2} = 11.86\ m/s$$

Since the velocity of the rock, V_{CHECK} (= 11.86 m/s) is greater than the minimum velocity, V_{MIN} (= 1.0 m/s), the rock is still considered to be moving. Since the rock is still moving, the simulation must continue for at least one more step.

Step 3

In a similar fashion to the previous step, the final rock conditions for step 2 are used as the initial conditions for step 3. The necessary parameters are determined and the quadratic equation is solved to find the time to intersection with the fourth slope segment:

$$q = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(20-22)}{(38-26)} = -0.1667$$

$$a = \frac{1}{2} g \cong -4.90$$

$$b = V_{Y0} - qV_{X0} = -10.19 - (-0.1667)6.06 \cong -9.185$$

$$c = Y_0 - Y_1 + q(X_1 - X_0) = 30.322 - 22 + (-0.1667)(26 - 22.764) \cong 7.782$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-9.185) \pm \sqrt{(-9.185)^2 - 4(-4.90)(7.782)}}{2(-4.90)} \cong -2.506 \; or \; 0.6332 \; s$$

t = -2.506 s is rejected because t must lie in the range [0,∞]. The intersection point and pre-impact velocity are determined:

$$X_I = V_{X0}t + X_0 = 6.06(0.6332) + 22.764 = 26.601 \; m$$

$$Y_I = \frac{1}{2} g t^2 + V_{Y0}t + Y_0 = \frac{1}{2}(-9.81)(0.6332)^2 + (-10.19)(0.6332) + 30.322 = 21.900 \; m$$

$$V_{XB} = V_{X0} = 6.06 \; m/s$$

$$V_{YB} = V_{Y0} + gt = -10.19 + (-9.81)0.6332 = -16.41 \; m/s$$

The velocities are transformed into components normal and tangential to the slope segment:

$$\theta = \tan^{-1}(q) = -9.4623^0$$

$$V_{NB} = (V_{YB})\cos(\theta) - (V_{XB})\sin(\theta) = (-16.41)\cos(-9.462) - (6.06)\sin(-9.462)$$
$$= -15.19\ m/s$$

$$V_{TB} = (V_{YB})\sin(\theta) + (V_{XB})\cos(\theta) = (-16.41)\sin(-9.462) + (6.06)\cos(-9.462)$$
$$= 8.67\ m/s$$

The friction function (F_1) and scaling function (F_2) are calculated:

$$F_1 = R_T + \frac{(1-R_T)}{\left(\frac{(V_{TB} - \omega_B \cdot r)}{C_{F1}}\right)^2 + 1.2} = 0.8 + \frac{(1-0.9)}{\left(\frac{(8.67 - 112.1 \cdot (0.104))}{6.096}\right)^2 + 1.2} = 0.969$$

$$F_2 = \frac{R_T}{[\frac{V_{NB}}{C_{F2} \cdot R_N}]^2 + 1} = \frac{0.9}{[\frac{-15.19}{76.2 \cdot 0.6}]^2 + 1} = 0.810$$

Calculate the outgoing velocities:

$$V_{NA} = R_N V_{NB} = 0.6(-15.19) = -9.112\ m/s$$

$$V_{TA} = \sqrt{\frac{r^2[I(\omega_B)^2 + m(V_{TB})^2] \cdot F_1 \cdot F_2}{I + m \cdot r^2}}$$
$$= \sqrt{\frac{(0.104)^2[0.0436(112.1)^2 + 10(8.67)^2] \cdot 0.969 \cdot 0.810}{0.0436 + 10 \cdot (0.104)^2}} = 8.541 m/s$$

$$\omega_A = \frac{V_{TA}}{r} = \frac{8.541}{0.104} = 81.83\, rad/s$$

The velocities are transformed back into vertical and horizontal components:

$$V_{XA} = (V_{NA})\sin(\theta) + (V_{TA})\cos(\theta) = (-9.112)\sin(-9.462) + (8.541)\cos(-9.462)$$
$$= 9.92\ m/s$$
$$V_{YA} = (V_{TA})\sin(\theta) - (V_{NA})\cos(\theta) = (8.541)\sin(-9.462) - (-9.112)\cos(-9.462)$$
$$= 7.58\ m/s$$

Step 3 is complete. The velocity of the rock, after impact, is calculated:

$$V_{CHECK} = \sqrt{V_{XA}^2 + V_{YA}^2} = \sqrt{(9.92)^2 + (7.58)^2} = 12.49\ m/s$$

Since the velocity of the rock, V_{CHECK} (=12.49 m/s) is greater than the minimum velocity, V_{MIN} (= 1.0 m/s), the rock is still considered to be moving. Since the rock is still moving, the simulation must continue for at least one more step.

Step 4

The final rock conditions for step 3 are used as the initial conditions for step 4. The necessary parameters are determined and the quadratic equation is solved to find the time to intersection with the sixth slope segment:

$$q = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(0-0)}{(89-46)} = 0$$

$$a = \frac{1}{2} g \cong -4.90$$

$$b = V_{Y0} - qV_{X0} = 7.58 - (0)9.92 \cong 7.584$$

$$c = Y_0 - Y_1 + q(X_1 - X_0) = 21.90 - 0 + 0(46 - 26.58) = 21.90$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(7.584) \pm \sqrt{(7.584)^2 - 4(-4.90)(21.90)}}{2(-4.90)} \cong -1.477\ or\ 3.023\ s$$

t = -1.477 s is rejected because t must lie in the range [0,∞]. The intersection point and pre-impact velocity are determined:

$$X_I = V_{X0}t + X_0 = 9.92(3.023) + 26.601 = 56.598\ m$$

$$Y_I = \frac{1}{2} gt^2 + V_{Y0}t + Y_0 = \frac{1}{2}(-9.81)(3.023)^2 + 7.58(3.023) + 21.900 = 0.000\ m$$

$$V_{XB} = V_{X0} = 9.92\ m/s$$

$$V_{YB} = V_{Y0} + gt = 7.58 + (-9.81)3.023 = -22.07\ m/s$$

The velocities are transformed into components normal and tangential to the slope segment:

$$\theta = \tan^{-1}(q) = 0.0^0$$

$$V_{NB} = (V_{YB})\cos(\theta) - (V_{XB})\sin(\theta) = (-22.07)\cos(0) - (9.92)\sin(0) = -22.07\ m/s$$

$$V_{TB} = (V_{YB})\sin(\theta) + (V_{XB})\cos(\theta) = (-22.07)\sin(0) + (9.92)\cos(0) = 9.92\,m/s$$

The friction function (F_1) and scaling function (F_2) are calculated:

$$F_1 = R_T + \frac{(1-R_T)}{\left(\dfrac{(V_{TB}-\omega_B \cdot r)}{C_{F1}}\right)^2 + 1.2} = 0.8 + \frac{(1-0.6)}{\left(\dfrac{(9.92-81.83\cdot(0.104))}{6.096}\right)^2 + 1.2} = 0.9197$$

$$F_2 = \frac{R_T}{[\dfrac{V_{NB}}{C_{F2}\cdot R_N}]^2 + 1} = \frac{0.6}{[\dfrac{-22.07}{76.2\cdot 0.4}]^2 + 1} = 0.3936$$

Calculate the outgoing velocities:

$$V_{NA} = R_N V_{NB} = 0.5(-22.07) = -8.83\ m/s$$

$$V_{TA} = \sqrt{\frac{r^2[I(\omega_B)^2 + m(V_{TB})^2]\cdot F_1 \cdot F_2}{I + m\cdot r^2}}$$

$$= \sqrt{\frac{(0.104)^2[0.0436(81.83)^2 + 10(9.92)^2]\cdot 0.9197\cdot 0.3936}{0.0436 + 10\cdot(0.104)^2}} = 5.74\,m/s$$

$$\omega_A = \frac{V_{TA}}{r} = \frac{5.74}{0.104} = 55.04\,rad/s$$

The velocities are transformed back into vertical and horizontal components (using equations 4.17 and 4.18):

$$V_{XA} = (V_{NA})\sin(\theta) + (V_{TA})\cos(\theta) = (-8.83)\sin(0) + (5.74)\cos(0) = 5.74\ m/s$$

$$V_{YA} = (V_{TA})\sin(\theta) - (V_{NA})\cos(\theta) = (5.74)\sin(0) - (-8.83)\cos(0) = 8.83\ m/s$$

Step 4 is complete. The velocity of the rock, after impact, is calculated:

$$V_{CHECK} = \sqrt{V_{XA}^2 + V_{YA}^2} = \sqrt{(5.74)^2 + (8.83)^2} = 10.53\ m/s$$

Since the velocity of the rock, V_{CHECK} (= 10.53 m/s) is greater than the minimum velocity, V_{MIN} (= 1.0 m/s), the rock is still considered to be moving. Since the rock is still moving, the simulation must continue for at least one more step. However, the hand calculations will not continue because they are very similar to step 4, and will not provide much further verification, only repetition.

Conclusion

The same geometry and parameters were input into RocFall and a simulation was performed. The results from RocFall were compared to the manual calculations. The results from the manual calculations were identical to the RocFall results for all practical purposes. The impact locations calculated by hand agreed with the program results up to the third decimal place in all cases (i.e. less than 0.5 mm difference, everywhere). Therefore, the angular velocity algorithm seems to be working correctly.